

# THE NONLINEAR SCHRÖDINGER EQUATION GROUND STATES ON PRODUCT SPACES

SUSANNA TERRACINI, NIKOLAY TZVETKOV, AND NICOLA VISCIGLIA

**ABSTRACT.** We study the nature of the Nonlinear Schrödinger equation ground states on the product spaces  $\mathbb{R}^n \times M^k$ , where  $M^k$  is a compact Riemannian manifold. We prove that for small  $L^2$  masses the ground states coincide with the corresponding  $\mathbb{R}^n$  ground states. We also prove that above a critical mass the ground states have nontrivial  $M^k$  dependence. Finally, we address the Cauchy problem issue which transform the variational analysis to dynamical stability results.

**MSC:** 35Q55, 37K45. **Keywords:** NLS, stability of solitons, rigidity.

## 1. INTRODUCTION

Our goal here is to study the nature of the Nonlinear Schrödinger equation ground states when the problem is posed on the product spaces  $\mathbb{R}^n \times M^k$ , where  $M^k$  is a compact Riemannian manifold. We thus consider the following Cauchy problems

$$(1.1) \quad \begin{cases} i\partial_t u - \Delta_{x,y} u - u|u|^\alpha = 0, & (t, x, y) \in \mathbb{R} \times \mathbb{R}^n \times M_y^k \\ u(0, x, y) = \varphi(x, y) \end{cases}$$

where

$$\Delta_{x,y} = \sum_{j=1}^n \partial_{x_j}^2 + \Delta_y$$

and  $\Delta_y$  is the Laplace-Beltrami operator on  $M_y^k$ . Recall that the Laplace-Beltrami operator is defined in local coordinates as follows:

$$\frac{1}{\sqrt{\det(g_{i,j}(y))}} \partial_{y_i} \sqrt{\det(g_{i,j}(y))} g^{i,j}(y) \partial_{y_j}$$

where  $g^{i,j}(y) = (g_{i,j}(y))^{-1}$  and  $g_{i,j}(y)$  is the metric tensor.

We assume that  $0 < \alpha < 4/(n+k)$  which corresponds to  $L^2$  subcritical nonlinearity.

In this paper, we shall study the following two questions:

- the existence and stability of solitary waves for (1.1);
- the global well posedness of the Cauchy problem associated to (1.1).

The equation (1.1) has two (at least formal) conservation laws, the energy

$$(1.2) \quad \mathcal{E}_{n,M^k,\alpha}(u) = \int_{M_y^k} \int_{\mathbb{R}^n_x} \left( \frac{1}{2} |\nabla_{x,y} u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \right) dx dv_{M_y^k}$$

and the  $L^2$  mass,

$$(1.3) \quad \|u\|_{L^2(\mathbb{R}^n \times M^k)}^2 = \int_{M_y^k} \int_{\mathbb{R}^n_x} |u|^2 dx dv_{M_y^k}$$

Here we denote by  $dvol_{M_y^k}$  the volume form on  $M^k$ . Recall that in local coordinates it can be written as  $\sqrt{\det(g_{i,j}(y))}dy$ . Moreover the  $i$ -th component (in local coordinates) of the gradient  $(\nabla_y u(y))$  is

$$g^{i,j}(y)\partial_{y_j}u$$

One has the classical Gagliardo-Nirenberg inequality

$$(1.4) \quad \|u\|_{L^{2+\alpha}(\mathbb{R}^n \times M^k)}^{2+\alpha} \leq C \|u\|_{H^1(\mathbb{R}^n \times M^k)}^{\theta(\alpha)} \|u\|_{L^2(\mathbb{R}^n \times M^k)}^{2+\alpha-\theta(\alpha)}$$

where  $\theta(\alpha) = (n+k)\alpha/2$ . Thus  $\theta(\alpha) < 2$  under our assumption  $0 < \alpha < 4/(n+k)$ . This implies that the conservation laws (1.2) and (1.3) imply a control on the  $H^1$  norm which excludes a  $L^2$  self-focusing blow-up and thus one expects that (1.1) has a well-defined global dynamics. This problem seems quite delicate for a general  $M^k$ . However if we replace  $M^k$  with  $\mathbb{R}^k$  it is well-known (see [11], [4] and the references therein) that (1.1) has a global strong solution for every  $L^2(\mathbb{R}^{n+k})$  initial data.

Our argument to construct stable solutions to (1.1) follows the one proposed in [5]. Hence we shall look at the following minimization problems:

$$(1.5) \quad K_{n,M^k,\alpha}^\rho = \inf_{\substack{u \in H^1(\mathbb{R}^n \times M^k) \\ \|u\|_{L^2(\mathbb{R}^n \times M^k)} = \rho}} \mathcal{E}_{n,M^k,\alpha}(u)$$

and  $\mathcal{E}_{n,M^k,\alpha}(u)$  is defined in (1.2). In the sequel we shall use the following notation:

$$(1.6) \quad \mathcal{M}_{n,M^k,\alpha}^\rho = \{v \in H^1(\mathbb{R}^n \times M^k) \mid \|v\|_{L^2(\mathbb{R}^n \times M^k)} = \rho \text{ and } \mathcal{E}_{n,M^k,\alpha}(v) = K_{n,M^k,\alpha}^\rho\}$$

The first result we state concerns the compactness of minimizing sequences to (1.5).

**Theorem 1.1.** *Let  $M^k$  be a compact manifold and  $0 < \alpha < 4/(n+k)$ . Then we have the following:*

$$(1.7) \quad K_{n,M^k,\alpha}^\rho > -\infty \text{ and } \mathcal{M}_{n,M^k,\alpha}^\rho \neq \emptyset, \forall \rho > 0;$$

$$(1.8) \quad \forall u_j \in H^1(\mathbb{R}^n \times M^k) \text{ s.t. } \|u_j\|_{L^2(\mathbb{R}^n \times M^k)} = \rho, \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(u_j) = K_{n,M^k,\alpha}^\rho$$

$\exists$  a subsequence  $u_{j_l}$  and  $\tau_l \in \mathbb{R}_x^n$  s.t.  $u_{j_l}(x + \tau_l, y)$  converges in  $H^1(\mathbb{R}^n \times M^k)$ .

The proof of Theorem 1.1 is based on the concentration compactness principle and it will be given in the appendix. Also the following stability theorem follows from a standard argument, hence its classical proof will be recalled in the appendix.

**Theorem 1.2.** *Let  $\rho > 0$  be fixed and  $n, M^k, \alpha$  as in Theorem 1.1. Assume moreover that*

$$(1.9) \quad \text{the Cauchy problem (1.1) is globally well posed for any data } \varphi \in \mathcal{U}$$

where  $\mathcal{U}$  is a  $H^1(\mathbb{R}^n \times M^k)$ -neighborhood of  $\mathcal{M}_{n,M^k,\alpha}^\rho$ .

Then the set  $\mathcal{M}_{n,M^k,\alpha}^\rho$  is orbitally stable, i.e.:

$$\begin{aligned} & \forall \epsilon > 0 \exists \delta = \delta(\epsilon) > 0 \text{ s.t.} \\ & \varphi \in \mathcal{U}, \inf_{v \in \mathcal{M}_{n,M^k,\alpha}^\rho} \|\varphi - v\|_{H^1(\mathbb{R}^n \times M^k)} < \delta(\epsilon) \\ & \text{implies } \sup_{t \in \mathbb{R}} \left( \inf_{v \in \mathcal{M}_{n,M^k,\alpha}^\rho} \|u_\varphi(t) - v\|_{H^1(\mathbb{R}^n \times M^k)} \right) < \epsilon \end{aligned}$$

where  $u_\varphi(t, x, y)$  is the unique global solution to (1.1).

Let us emphasize that the stability result stated in Theorem 1.2 has two major defaults: the first one is that we don't have an explicit description of the minimizers  $\mathcal{M}_{n, M^k, \alpha}^\rho$ ; the second one is that it is subordinated to (1.9), i.e. the global well posedness of the Cauchy problem (1.1). The main contributions of this paper concern a partial understanding of the aforementioned questions.

Notice that (see [4]) a special family of solutions to (1.1) is given by

$$u(t, x, y) = e^{-i\omega t} u_{n, \omega, \alpha}(x)$$

where  $\omega > 0$  and  $u_{n, \omega, \alpha}(x)$  is defined as the unique radial solution to:

$$(1.10) \quad \begin{aligned} -\Delta_x u_{n, \omega, \alpha} + \omega u_{n, \omega, \alpha} &= u_{n, \omega, \alpha} |u_{n, \omega, \alpha}|^\alpha \\ u_{n, \omega, \alpha} &\in H^1(\mathbb{R}_x^n), \quad u_{n, \omega, \alpha}(x) > 0, \quad x \in \mathbb{R}_x^n \end{aligned}$$

Next, we set

$$(1.11) \quad \mathcal{N}_{n, \omega, \alpha} = \{e^{i\theta} u_{n, \omega, \alpha}(x + \tau) | \tau \in \mathbb{R}^n, \theta \in \mathbb{R}\}$$

Notice that there is a natural embedding  $H^1(\mathbb{R}_x^n) \subset H^1(\mathbb{R}_x^n \times M_y^k)$ . In fact every function in  $H^1(\mathbb{R}_x^n)$  can be extended in a trivial way w.r.t. the  $y$  variable on  $\mathbb{R}_x^n \times M_y^k$  and this extension will belong to  $H^1(\mathbb{R}^n \times M^k)$ . In particular since now on the set  $\mathcal{N}_{n, \omega, \alpha}$  defined in (1.11), will be considered without any further comment in a twofold way: as a subset of  $H^1(\mathbb{R}_x^n)$  and  $H^1(\mathbb{R}_x^n \times M_y^k)$ . By a rescaling argument one can prove that the function

$$(0, \infty) \ni \omega \rightarrow \|u_{n, \omega, \alpha}\|_{L^2(\mathbb{R}_x^n)}^2 \in (0, \infty)$$

is strictly increasing for any  $0 < \alpha < \frac{4}{n}$  and

$$\lim_{\omega \rightarrow \infty} \|u_{n, \omega, \alpha}\|_{L^2(\mathbb{R}_x^n)} = \infty \quad \text{and} \quad \lim_{\omega \rightarrow 0} \|u_{n, \omega, \alpha}\|_{L^2(\mathbb{R}_x^n)} = 0$$

As a consequence for any fixed  $0 < \alpha < \frac{4}{n}$  we have:

$$(1.12) \quad \forall \rho > 0 \exists! \omega(\rho) > 0 \text{ s.t. } \|u_{n, \omega(\rho), \alpha}\|_{L^2(\mathbb{R}_x^n)} = \rho$$

In next theorem the set  $\mathcal{N}_{n, \omega, \alpha}$  is the one defined in (1.11) and  $\mathcal{M}_{n, M^k, \alpha}^\rho$  is defined in (1.6).

**Theorem 1.3.** *Let  $n, M^k, \alpha$  as in Theorem 1.2. There exists  $\rho^* \in (0, \infty)$  such that:*

$$(1.13) \quad \mathcal{M}_{n, M^k, \alpha}^\rho = \mathcal{N}_{n, \omega(\rho/\sqrt{\text{vol}(M^k)}), \alpha}, \quad \forall \rho < \rho^*$$

and

$$(1.14) \quad \mathcal{M}_{n, M^k, \alpha}^\rho \cap \mathcal{N}_{n, \omega(\rho/\sqrt{\text{vol}(M^k)}), \alpha} = \emptyset, \quad \forall \rho > \rho^*$$

where  $\omega(\rho/\sqrt{\text{vol}(M^k)})$  is uniquely defined in (1.12). In particular for  $\rho > \rho^*$  the elements of  $\mathcal{M}_{n, M^k, \alpha}^\rho$  depend in a nontrivial way on the  $M^k$  variable.

By the approach of Weinstein [13] one may expect that  $\mathcal{N}_{n, \omega, \alpha}$  is stable under (1.1) for  $\alpha < 4/n$  and  $\omega$  small enough, see [9] for a recent related work. It should however be pointed out that in such a stability result one would not get the variational description of  $\mathcal{N}_{n, \omega, \alpha}$  as is the case in Theorem 1.3 ( $\alpha < 4/(n+k)$ ). We underline that by combining Theorem 1.2 and Theorem 1.3 we get a stable set for large values of the mass  $\rho$ , and in general it is independent of the solitary solitary

waves associated to NLS in  $\mathbb{R}^n$ .

Next we shall focus on the question of the global well-posedness of the Cauchy problem associated to (1.1) in the particular case  $n \geq 1$ ,  $k = 1$ . For every  $n > 1$  we fix the numbers

$$p := p(n, \alpha) = \frac{4(2 + \alpha)}{n\alpha} \text{ and } q := q(n, \alpha) = 2 + \alpha$$

and for every  $T > 0$  we define the localized norms:

$$(1.15) \quad \|u(t, x, y)\|_{X_T} \equiv \|u(t, x, y)\|_{L^p((-T, T); L^q(\mathbb{R}_x^n; H^1(M_y^1)))}$$

and

$$(1.16) \quad \|u(t, x, y)\|_{Y_T} \equiv \|\nabla_x u\|_{L^p((-T, T); L^q(\mathbb{R}_x^n; L^2(M_y^1)))}$$

**Theorem 1.4.** *Let  $n \geq 1$  be fixed and  $\alpha < 4/(n + 1)$ , then for every initial data  $\varphi \in H^1(\mathbb{R}^n \times M^1)$ , the Cauchy problem (1.1) has a unique global solution  $u(t, x, y)$  satisfying :*

$$u(t, x, y) \in \mathcal{C}((-T, T); H^1(\mathbb{R}^n \times M^1)) \cap X_T \cap Y_T, \quad \forall T > 0$$

*Remark 1.1.* The main difficulty in the analysis of the Cauchy problem (1.1) (compared with the Cauchy problem in the euclidean space) is related with the fact that the propagator  $e^{-it\Delta_{x,y}}$  on  $\mathbb{R}^n \times M_y^1$  does not satisfies the Strichartz estimates which are available for the propagator  $e^{-it\Delta_{\mathbb{R}^{n+k}}}$  on the euclidean space  $\mathbb{R}^{n+k}$ .

Let us now describe some other known cases when (1.1) is well-posed in  $H^1(\mathbb{R}^n \times M^k)$  under the assumption  $\alpha < 4/(n + k)$ . Using the analysis of [2, 3] one may prove such a well-posedness result in the case  $\mathbb{R} \times M^2$ , i.e.  $n = 1$  and  $k = 2$ . Moreover, using the analysis of the recent papers [6] and [7] one may also prove such a well-posedness result in the cases  $\mathbb{R}^2 \times \mathbb{T}^2$  and  $\mathbb{R} \times \mathbb{T}^3$  respectively.

*Notation.* Next we fix some notations. We denote by  $L_x^p$  and  $H_x^s$  respectively the space  $L^p(\mathbb{R}_x^n)$  and  $H^s(\mathbb{R}_x^n)$ . We also use the notation  $L_{x,y}^p = L^p(\mathbb{R}_x^n \times M_y^k)$  and  $L_x^p L_y^q = L^p(\mathbb{R}_x^n; L^q(M_y^k))$ . If  $v(t)$  is a time dependent function defined on  $\mathbb{R}_t$  and valued in a Banach space  $X$ , then we define

$$\|v\|_{L_t^p(X)}^p = \int_{\mathbb{R}} \|v(t)\|_X^p dt$$

For every  $p \in [1, \infty]$  we denote by  $p' \in [1, \infty]$  its conjugate Hölder exponent. We denote by  $e^{-it\Delta_{x,y}}$  the free propagator associated to the Schrödinger equation on  $\mathbb{R}_x^n \times M_y^k$ .

## 2. SOME USEFUL RESULTS ON THE EUCLIDEAN SPACE $\mathbb{R}_x^n$ WITH $n \geq 1$

In this section we recall some well known facts (see [4]) related to the following minimization problem on  $\mathbb{R}_x^n$ :

$$(2.1) \quad I_{n,\alpha}^\rho = \inf_{\substack{u \in H_x^1 \\ \|u\|_{L_x^2} = \rho}} \mathcal{E}_{n,\alpha}(u)$$

where for  $\alpha < 4/n$

$$(2.2) \quad \mathcal{E}_{n,\alpha}(u) = \frac{1}{2} \int_{\mathbb{R}_x^n} |\nabla_x u|^2 dx - \frac{1}{2 + \alpha} \int_{\mathbb{R}_x^n} |u|^{2+\alpha} dx$$

By an elementary rescaling argument we have

$$(2.3) \quad I_{n,\alpha}^\rho = \rho^{(8+4\alpha-2\alpha n)/(4-\alpha n)} I_{n,\alpha}^1$$

It is well-known that

$$(2.4) \quad -\infty < I_{n,\alpha}^\rho < 0, \quad \forall \rho > 0$$

and

$$(2.5) \quad \mathcal{M}_{n,\alpha}^\rho = \mathcal{N}_{n,\omega(\rho),\alpha}$$

where  $\mathcal{N}_{n,\omega,\alpha}$  is defined in (1.11),

$$(2.6) \quad \mathcal{M}_{n,\alpha}^\rho = \{u \in H_x^1 \mid \|u\|_{L_x^2} = \rho \text{ and } \mathcal{E}_{n,\alpha}(u) = I_{n,\alpha}^\rho\}$$

and  $\omega(\rho)$  is defined uniquely (see (1.12)) by the relation

$$\|u_{n,\omega(\rho),\alpha}\|_{L_x^2} = \rho$$

We also recall that the functions  $u_{n,\omega,\alpha}$  (defined as the unique radially symmetric and positive solution to (1.10)) satisfy the following Pohozaev type identity (for a proof of (2.7) see the proof of (3.21) in next section):

$$(2.7) \quad \int_{\mathbb{R}_x^n} |\nabla_x u_{n,\omega,\alpha}|^2 dx = \frac{\alpha n}{2(\alpha+2)} \int_{\mathbb{R}_x^n} |u_{n,\omega,\alpha}|^{2+\alpha} dx$$

On the other hand if we multiply (1.10) by  $u_{n,\omega,\alpha}$  and we integrate by parts then we get

$$\int_{\mathbb{R}_x^n} |\nabla_x u_{n,\omega,\alpha}|^2 dx + \omega \|u_{n,\omega,\alpha}\|_{L_x^2}^2 = \int_{\mathbb{R}_x^n} |u_{n,\omega,\alpha}|^{2+\alpha} dx$$

that in conjunction with (2.7) gives

$$(2.8) \quad \begin{aligned} \omega \|u_{n,\omega,\alpha}\|_2^2 &= \frac{2\alpha+4-\alpha n}{\alpha n} \int_{\mathbb{R}_x^n} |\nabla_x u_{n,\omega,\alpha}|^2 dx \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n-4} \left( \frac{1}{2} \int_{\mathbb{R}_x^n} |\nabla_x u_{n,\omega,\alpha}|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}_x^n} |u_{n,\omega,\alpha}|^{2+\alpha} dx \right) \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n-4} I_{n,\alpha}^{\|u_{n,\omega,\alpha}\|_{L_x^2}} \end{aligned}$$

(at the last step we have used the fact that due to (2.5) we have that  $u_{n,\omega,\alpha}$  is a minimizer for  $\mathcal{E}_{n,\alpha}$  on its associated constrained).

Finally notice that by (2.7) we deduce

$$(2.9) \quad I_{n,\alpha}^{\|u_{n,\omega,\alpha}\|_{L_x^2}} = \mathcal{E}_{n,\alpha}(u_{n,\omega,\alpha}) = \frac{\alpha n-4}{2\alpha n} \int_{\mathbb{R}_x^n} |\nabla_x u_{n,\omega,\alpha}|^2 dx$$

### 3. AN AUXILIARY PROBLEM

In this section we study the minimizers of the following minimization problems

$$(3.1) \quad J_{n,M^k,\alpha,\lambda} = \inf_{\substack{u \in H^1(\mathbb{R}^n \times M^k) \\ \|u\|_{L_{x,y}^2} = 1}} \mathcal{E}_{n,M^k,\alpha,\lambda}(u)$$

where

$$\mathcal{E}_{n,M^k,\alpha,\lambda}(u) = \int_{M_y^k} \int_{\mathbb{R}_x^n} \left( \frac{\lambda}{2} |\nabla_y u|^2 + \frac{1}{2} |\nabla_x u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \right) dx d\text{vol}_{M_y^k}$$

We also introduce the following sets:

$$\mathcal{M}_{n,M^k,\alpha,\lambda} = \{w \in H^1(\mathbb{R}^n \times M^k) \mid \|w\|_{L^2_{x,y}} = 1 \text{ and } \mathcal{E}_{n,M^k,\alpha,\lambda}(w) = J_{n,M^k,\alpha,\lambda}\}$$

**Theorem 3.1.** *Let  $n, M^k$  and  $0 < \alpha < \frac{4}{n+k}$  be given. There exists  $\lambda^* \in (0, \infty)$  such that:*

$$(3.2) \quad \mathcal{M}_{n,M^k,\alpha,\lambda} = \mathcal{N}_{n,\bar{\omega},\alpha}, \quad \forall \lambda > \lambda^*$$

and

$$(3.3) \quad \mathcal{M}_{n,M^k,\alpha,\lambda} \cap \mathcal{N}_{n,\bar{\omega},\alpha} = \emptyset, \quad \forall \lambda < \lambda^*$$

where  $\bar{\omega}$  is defined by the condition

$$\text{vol}(M^k) \|u_{n,\bar{\omega},\alpha}\|_{L^2_x}^2 = 1$$

We fix a sequence  $\lambda_j \rightarrow \infty$  and a corresponding sequence of functions  $u_{\lambda_j} \in \mathcal{M}_{n,M^k,\alpha,\lambda_j}$ . In the sequel we shall assume that

$$(3.4) \quad u_{\lambda_j}(x, y) \geq 0, \quad \forall (x, y) \in \mathbb{R}^n_x \times M^k_y$$

Indeed it is well-known that if  $u_{\lambda_j}$  is a minimizer, then also  $|u_{\lambda_j}|$  is a minimizer. In particular there exists at least one minimizer which satisfies (3.4).

Notice that the functions  $u_{\lambda_j}$  depend in principle on the full set of variables  $(x, y)$ . Our aim is to prove that for  $j$  large and up to subsequence, the functions  $u_{\lambda_j}$  will not depend explicitly on the variable  $y$ .

First we prove some a priori bounds satisfied by  $u_{\lambda_j}(x, y)$ . Recall that the quantities  $I_{n,\alpha}^\rho$  are defined in (2.1).

**Lemma 3.1.** *Assume the same assumptions as in Theorem 3.1, then we have:*

$$(3.5) \quad \lim_{j \rightarrow \infty} J_{n,M^k,\alpha,\lambda_j} = \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$$

and

$$(3.6) \quad \lim_{j \rightarrow \infty} \lambda_j \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_y u_{\lambda_j}|^2 dx d\text{vol}_{M^k_y} = 0$$

**Proof.** First notice that

$$(3.7) \quad J_{n,M^k,\alpha,\lambda_j} \leq \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$$

In fact let  $w(x) \in H^1_x$  be such that  $\|w\|_{L^2_x} = \frac{1}{\sqrt{\text{vol}(M^k)}}$  and  $\mathcal{E}_{n,\alpha}(w) = I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$ .

Then we get easily:

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} &\leq \mathcal{E}_{n,M^k,\alpha,\lambda_j}(w(x)) \\ &= \text{vol}(M^k) \left( \frac{1}{2} \int_{\mathbb{R}^n_x} |\nabla_x w|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^n_x} |w|^{2+\alpha} dx \right) = \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}} \end{aligned}$$

and this concluded the proof of (3.7).

Next we claim that

$$(3.8) \quad \lim_{j \rightarrow \infty} \int_{M^k_y} \int_{\mathbb{R}^n_x} |\nabla_y u_{\lambda_j}|^2 dx d\text{vol}_{M^k_y} = 0$$

In order to prove this fact assume by the absurd that it is false then there exists a subsequence of  $\lambda_j$  (that we still denote by  $\lambda_j$ ) such that

$$\lim_{j \rightarrow \infty} \lambda_j = \infty \text{ and } \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx dv_{M_y^k} \geq \epsilon_0 > 0$$

and in particular

$$(3.9) \quad \lim_{j \rightarrow \infty} (\lambda_j - 1) \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx dv_{M_y^k} = \infty$$

On the other hand by the classical Gagliardo Nirenberg inequality (see (1.4)) we deduce the existence of  $0 < \mu < 2$  such that:

$$\begin{aligned} & \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) dx dv_{M_y^k} - \frac{1}{2 + \alpha} \int_{M_y^k} \int_{\mathbb{R}_x^n} |v|^{2+\alpha} dx dv_{M_y^k} \\ & \geq \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) dx dv_{M_y^k} \\ & - C \left[ \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) dx dv_{M_y^k} \right]^\mu \\ & \geq \inf_{t > 0} (1/2 t^2 - C t^\mu) = C(\mu) > -\infty \\ & \forall v \in H^1(\mathbb{R}^n \times M^k) \text{ s.t. } \|v\|_{L_{x,y}^2} = 1 \end{aligned}$$

By the previous inequality we get

$$\begin{aligned} & \mathcal{E}_{n,M^k,\alpha,\lambda_j}(v) - \frac{1}{2}(\lambda_j - 1) \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y v|^2 \geq -\frac{1}{2} + C(\mu) \\ & \forall v \in H^1(\mathbb{R}^n \times M^k) \text{ s.t. } \|v\|_{L_{x,y}^2} = 1 \end{aligned}$$

In particular if we choose  $v = u_{\lambda_j}$  then we get

$$\begin{aligned} & J_{n,M^k,\alpha,\lambda_j} = \mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \\ & \geq \frac{1}{2}(\lambda_j - 1) \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx dv_{M_y^k} - \frac{1}{2} + C(\mu) \end{aligned}$$

By (3.9) this implies  $\lim_{n \rightarrow \infty} J_{n,M^k,\alpha,\lambda_j} = \infty$  and this is in contradiction with (3.7). Hence (3.8) is proved.

Next we introduce the functions

$$w_j(y) = \|u_{\lambda_j}(x, y)\|_{L_x^2}^2$$

Notice that

$$(3.10) \quad \|w_j(y)\|_{L_y^1} = 1$$

and moreover

$$\begin{aligned} & \int_{M_y^k} |\nabla_y w_j(y)| dv_{M_y^k} \leq C \int_{M_y^k} \int_{\mathbb{R}_x^n} |u_{\lambda_j}(x, y)| |\nabla_y u_{\lambda_j}(x, y)| dx dv_{M_y^k} \\ & \leq C \|u_{\lambda_j}\|_{L_{x,y}^2} \|\nabla_y u_{\lambda_j}\|_{L_{x,y}^2} \end{aligned}$$

Hence due to (3.8) we get

$$(3.11) \quad \lim_{j \rightarrow \infty} \|\nabla_y w_j\|_{L_y^1} = 0$$

By combining (3.10) and (3.11) with the Rellich compactness theorem and with the Sobolev embedding  $W^{1,1}(M^1) \subset L^\infty(M^1)$  and  $W^{1,1}(M^2) \subset L^2(M^2)$  we deduce respectively in the case  $k = 1$  and  $k = 2$  that (up to a subsequence)

$$(3.12) \quad \lim_{j \rightarrow \infty} \|w_j(y) - 1/\text{vol}(M^1)\|_{L_y^r} = 0, \quad \forall 1 \leq r < \infty$$

and

$$(3.13) \quad \lim_{j \rightarrow \infty} \|w_j(y) - 1/\text{vol}(M^2)\|_{L_y^r} = 0, \quad \forall 1 \leq r < 2$$

For  $k > 2$  we use the Sobolev embedding  $H^1(M^k) \subset L^{2k/(k-2)}(M^k)$  and we get

$$\sup_j \|u_{\lambda_j}\|_{L_x^2 L_y^{2k/(k-2)}} \leq C \sup_j \|u_{\lambda_j}\|_{L_x^2 H^1(M_y^k)} < \infty$$

(where at the last step we have used the fact  $\sup_j (\|u_{\lambda_j}\|_{L_{x,y}^2} + \|\nabla_y u_{\lambda_j}\|_{L_{x,y}^2}) < \infty$ ). By the Minkowski inequality the bound above implies  $\sup_j \|u_{\lambda_j}\|_{L_y^{2k/(k-2)} L_x^2}$  which is equivalent to the condition

$$(3.14) \quad \sup_j \|w_j(y)\|_{L_y^{k/(k-2)}} < \infty \text{ for } k > 2$$

By combining (3.10) and (3.11) with the Rellich compactness theorem we deduce that up to a subsequence

$$\|w_j(y) - 1/\text{vol}(M^k)\|_{L_y^1} = 0 \text{ for } k > 2$$

and hence by interpolation with (3.14) we get

$$(3.15) \quad \|w_j(y) - 1/\text{vol}(M^k)\|_{L_y^r} = 0 \text{ for } k > 2, 1 \leq r < k/(k-2)$$

By the definition of  $I_{n,\alpha}^\rho$  (see (2.1)) and (2.3) we get

$$(3.16) \quad \begin{aligned} & \frac{1}{2} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}(x, y)|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}_x^n} |u_{\lambda_j}(x, y)|^{2+\alpha} dx \\ & \geq I_{n,\alpha}^{\|u_{\lambda_j}(\cdot, y)\|_{L_x^2}} = I_{n,\alpha}^1 \|u_{\lambda_j}(\cdot, y)\|_{L_x^2}^{(8+4\alpha-2\alpha n)/(4-\alpha n)} = I_{n,\alpha}^1 w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)} \\ & \quad \forall y \in M^k, \quad \forall j \in \mathbb{N} \end{aligned}$$

Next notice that by definition

$$(3.17) \quad \begin{aligned} J_{n,M^k,\alpha,\lambda_j} &= \mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \\ &= \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} (\lambda_j |\nabla_y u_{\lambda_j}|^2 + |\nabla_x u_{\lambda_j}|^2) dx dy - \frac{1}{2+\alpha} \int_{M_y^k} \int_{\mathbb{R}_x^n} |u|^{2+\alpha} dx d\text{vol}_{M_y^k} \end{aligned}$$

and we can continue

$$(3.18) \quad \begin{aligned} \dots & \geq \int_{M_y^k} \left( \frac{1}{2} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}(x, y)|^2 dx - \frac{1}{2+\alpha} \int_{\mathbb{R}_x^n} |u_{\lambda_j}(x, y)|^{2+\alpha} dx \right) d\text{vol}_{M_y^k} \\ & \geq I_{n,\alpha}^1 \int_{M_y^k} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)} d\text{vol}_{M_y^k} \\ & = I_{n,\alpha}^1 \text{vol}(M^k) \text{vol}(M^k)^{-(4+2\alpha-\alpha n)/(4-\alpha n)} + o(1) \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $j \rightarrow \infty$  and at the last step we have combined (3.12), (3.13) and (3.15) respectively for  $k = 1$ ,  $k = 2$  and  $k > 2$  and we used our assumption on  $\alpha$ . By combining this fact with (2.3) we have

$$(3.19) \quad \liminf_{j \rightarrow \infty} J_{n,M^k,\alpha,\lambda_j} \geq \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$$



Hence (3.5) follows by combining (3.7) with (3.19).

Next we prove (3.6). For that purpose, it suffices to keep the term  $\lambda_j |\nabla_y u_{\lambda_j}|^2$  in the previous analysis. Namely, by combining (3.5) with (3.17) and (3.18) we get

$$(3.20) \quad \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}} + g(j) \geq \frac{1}{2} \lambda_j \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx d\text{vol}_{M_y^k} + h(j)$$

where

$$\lim_{j \rightarrow \infty} g(j) = 0$$

and

$$\liminf_{j \rightarrow \infty} h(j) \geq \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$$

Hence (3.6) follows by (3.20). □

**Lemma 3.2.** *We have the following identity:*

$$(3.21) \quad \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx d\text{vol}_{M_y^k} = \frac{\alpha n}{2(2+\alpha)} \int_{M_y^k} \int_{\mathbb{R}_x^n} |u_{\lambda_j}|^{2+\alpha} dx d\text{vol}_{M_y^k}$$

Moreover there exist  $J \in \mathbb{N}$  such that

$$\forall j > J \quad \exists \omega(\lambda_j) > 0 \text{ s.t.}$$

$$(3.22) \quad -\lambda_j \Delta_y u_{\lambda_j} - \Delta_x u_{\lambda_j} + \omega(\lambda_j) u_{\lambda_j} = u_{\lambda_j} |u_{\lambda_j}|^\alpha$$

and the following limit exists

$$(3.23) \quad \lim_{j \rightarrow \infty} \omega(\lambda_j) = \bar{\omega} \in (0, \infty)$$

**Proof.** Since  $u_{\lambda_j}$  is a constrained minimizer for  $\mathcal{E}_{n,M^k,\alpha,\lambda_j}$  on the ball of size 1 in  $L^2(\mathbb{R}^n \times M^k)$ , then we get

$$\frac{d}{d\epsilon} \left[ \mathcal{E}_{n,M^k,\alpha,\lambda_j}(\epsilon^{\frac{n}{2}} u_{\lambda_j}(\epsilon x, y)) \right]_{\epsilon=1} = 0$$

which is equivalent to

$$\begin{aligned} & \frac{d}{d\epsilon} \left[ \frac{1}{2} \lambda_j \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx d\text{vol}_{M_y^k} \right. \\ & \left. + \frac{1}{2} \epsilon^2 \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx d\text{vol}_{M_y^k} - \frac{1}{2+\alpha} \epsilon^{\alpha n/2} \|u_{\lambda_j}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \right]_{\epsilon=1} = 0 \end{aligned}$$

By computing explicitly the derivative (in  $\epsilon$ ) we deduce (3.21).

Next notice that by using the Lagrange multiplier technique we get (3.22) for a suitable  $\omega(\lambda_j) \in \mathbb{R}$ . On the other hand by (3.22) we get

$$\begin{aligned} & \int_{M_y^k} \int_{\mathbb{R}_x^n} (\lambda_j |\nabla_y u_{\lambda_j}|^2 + |\nabla_x u_{\lambda_j}|^2) dx d\text{vol}_{M_y^k} + \omega(\lambda_j) \|u_{\lambda_j}\|_{L_{x,y}^2}^2 \\ & = \int_{M_y^k} \int_{\mathbb{R}_x^n} |u_{\lambda_j}|^{2+\alpha} dx d\text{vol}_{M_y^k} \end{aligned}$$

that by (3.21) gives

$$\omega(\lambda_j) = \frac{-\alpha n + 4 + 2\alpha}{\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} - \lambda_j \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 dx dvol_{M_y^k}$$

and hence by (3.6) we get

$$(3.24) \quad \omega(\lambda_j) = \frac{-\alpha n + 4 + 2\alpha}{\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} + o(1)$$

where  $\lim_{j \rightarrow \infty} o(1) = 0$ .

On the other hand notice that by (3.21) we get

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} &= \mathcal{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \\ &= \frac{\alpha n - 4}{2\alpha n} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} + \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} \lambda_j |\nabla_y u_{\lambda_j}|^2 dx dvol_{M_y^k} \end{aligned}$$

and by (3.6)

$$(3.25) \quad \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} = \frac{2\alpha n}{\alpha n - 4} J_{n,M^k,\alpha,\lambda_j} + o(1)$$

By (3.5) it implies

$$(3.26) \quad \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u_{\lambda_j}|^2 dx dvol_{M_y^k} = \frac{2\alpha n}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} + o(1)$$

that in conjunction with (3.24) and (2.4) implies  $\omega(\lambda_j) > 0$  for  $j$  large enough. Moreover (3.23) follows by (3.24) and (3.26).  $\square$

Next recall that the sets  $\mathcal{M}_{n,\alpha}^\rho$  are the ones defined in (2.6).

**Lemma 3.3.** *Let  $\bar{\omega}$  be as in (3.23) and let  $v(x) \in \mathcal{M}_{n,\alpha}^{1/\sqrt{vol(M^k)}}$  be such that  $v(x) > 0$ . Then*

$$-\Delta_x v + \bar{\omega} v = v|v|^\alpha$$

**Proof.** It is well-known that

$$-\Delta_x v + \omega_1 v = v|v|^\alpha$$

for a suitable  $\omega_1 > 0$ . More precisely we can assume that up to translation  $v = u_{n,\omega_1,\alpha}$ . Our aim is to prove that  $\omega_1 = \bar{\omega}$ . Notice that by (2.8)

$$(3.27) \quad \omega_1 \frac{1}{vol(M^k)} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{\|v\|_{L_x^2}} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

On the other hand by (3.24) and (3.26) we get

$$\omega(\lambda_j) = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}} + o(1)$$

and hence passing to the limit in  $j$  we get

$$(3.28) \quad \bar{\omega} = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} vol(M^k) I_{n,\alpha}^{1/\sqrt{vol(M^k)}}$$

By combining (3.27) and (3.28) we get  $\bar{\omega} = \omega_1$ .  $\square$

**Lemma 3.4.** *There exist a subsequence of  $\lambda_j$  (that we shall denote still by  $\lambda_j$ ) and a sequence  $\tau_j \in \mathbb{R}_x^n$  such that*

$$\lim_{j \rightarrow \infty} \|u_{\lambda_j}(x + \tau_j, y) - u_{\bar{\omega}}\|_{H^1(\mathbb{R}^n \times M^k)} = 0$$

where  $u_{\bar{\omega}} \in \mathcal{N}_{n, \bar{\omega}, \alpha}$ ,  $u_{\bar{\omega}} > 0$  and  $\bar{\omega}$  is defined in (3.23).

**Proof.** By combining (3.6) and (3.26), and since  $\|u_{\lambda_j}\|_{L_{x,y}^2} = 1$ , we deduce that  $u_{\lambda_j}$  is bounded in  $H^1(\mathbb{R}^n \times M^k)$ . Moreover by combining (3.5) with the fact that  $I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}} < 0$  (see (2.4)) then we get

$$\inf_j \|u_{\lambda_j}\|_{L_{x,y}^{2+\alpha}} > 0$$

By using the localized version of the Gagliardo Nirenberg inequality (6.5) (in the same spirit as in the appendix) we get the existence (up to subsequence) of  $\tau_j \in \mathbb{R}_x^n$  such that

$$u_{\lambda_j}(x + \tau_j, y) \rightharpoonup w \neq 0 \text{ in } H^1(\mathbb{R}^n \times M^k)$$

Moreover due to (3.4) we can assume that

$$w(x, y) \geq 0 \text{ a.e. } (x, y) \in \mathbb{R}_x^n \times M_y^k$$

and by (3.6) we get  $\nabla_y w = 0$ . In particular  $w$  is  $y$ -independent.

By combining (3.6) and (3.23) we pass to the limit in (3.22) in the distribution sense and we get

$$(3.29) \quad -\Delta_x w + \bar{\omega} w = w|w|^\alpha \text{ in } \mathbb{R}_x^n, \quad w(x) \geq 0, \quad w \neq 0$$

We claim that

$$(3.30) \quad \|w\|_{L_x^2} = \frac{1}{\sqrt{\text{vol}(M^k)}}$$

If not then we can assume  $\|w\|_{L_x^2} = \beta < \frac{1}{\sqrt{\text{vol}(M^k)}}$  and since  $w$  solves (3.29) by (2.5) we get

$$(3.31) \quad w \in \mathcal{M}_{n,\alpha}^\beta$$

On the other hand by Lemma 3.3 the equation (3.29) is satisfied by any  $v \in \mathcal{M}_{n,\alpha}^{\frac{1}{\sqrt{\text{vol}(M^k)}}}$ . Hence again by (2.5) and by the injectivity of the map  $\rho \rightarrow \omega(\rho)$  (see (1.12)) we deduce that necessarily  $\beta = \frac{1}{\sqrt{\text{vol}(M^k)}}$ .

In particular by (3.30) we deduce

$$\lim_{j \rightarrow \infty} \|u_{\lambda_j}(x + \tau_j, y) - w\|_{L_{x,y}^2} = 0$$

Next notice that by (3.6) and since we have already proved that  $\nabla_y w = 0$  we deduce that

$$\lim_{j \rightarrow \infty} \|\nabla_y u_{\lambda_j}(x + \tau_j, y)\|_{L_{x,y}^2} = 0 = \|\nabla_y w\|_{L_{x,y}^2}$$

Hence in order to conclude that  $u_{\lambda_j}(x + \tau_j, y)$  converges strongly to  $w$  in  $H^1(\mathbb{R}^n \times M^k)$  it is sufficient to prove that

$$\lim_{j \rightarrow \infty} \|\nabla_x u_{\lambda_j}(x + \tau_j, y)\|_{L_{x,y}^2} = \sqrt{\text{vol}(M^k)} \|\nabla_x w\|_{L_x^2} = \|\nabla_x w\|_{L_{x,y}^2}$$

This last fact follows by combining (2.9) (where we use the fact that  $w \in \mathcal{N}_{n, \bar{\omega}, \alpha}$  by (3.29) and  $\|w\|_{L_x^2} = \frac{1}{\sqrt{\text{vol}(M^k)}}$  by (3.30)) and (3.26).

□

**Lemma 3.5.** *There exists  $j_0 > 0$  such that*

$$\nabla_y u_{\lambda_j} = 0, \quad \forall j > j_0$$

**Proof.** By Lemma 3.4 we can assume that

$$(3.32) \quad u_{\lambda_j} \rightarrow u_{\bar{\omega}} \text{ in } H^1(\mathbb{R}^n \times M^k)$$

We introduce  $w_j = \sqrt{-\Delta_y} u_{\lambda_j}$ . Notice that due to (3.22) the functions  $w_j$  satisfy

$$(3.33) \quad -\lambda_j \Delta_y w_j - \Delta_x w_j + \omega(\lambda_j) w_j = \sqrt{-\Delta_y} (u_{\lambda_j} |u_{\lambda_j}|^\alpha)$$

that after multiplication by  $w_j$  implies

$$(3.34) \quad \int_{M_y^k} \int_{\mathbb{R}_x^n} \left[ \lambda_j |\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \omega(\lambda_j) |w_j|^2 - \sqrt{-\Delta_y} (u_{\lambda_j} |u_{\lambda_j}|^\alpha) w_j \right] dx dv_{M_y^k} = 0$$

In turn it gives

$$(3.35) \quad 0 = \int_{M_y^k} \int_{\mathbb{R}_x^n} (\lambda_j - 1) |\nabla_y w_j|^2 - (\alpha + 1) \sqrt{-\Delta_y} (u_{\lambda_j} |u_{\bar{\omega}}|^\alpha) w_j dx dv_{M_y^k} + \\ \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \bar{\omega} |w_j|^2 + \sqrt{-\Delta_y} (u_{\lambda_j} ((\alpha + 1) |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j dx dv_{M_y^k} \\ + \int_{M_y^k} \int_{\mathbb{R}_x^n} (\omega(\lambda_j) - \bar{\omega}) |w_j|^2 dx dy \equiv \\ I_j + II_j + III_j$$

Next we fix an orthonormal basis of eigenfunctions for  $-\Delta_y$ , i.e.  $-\Delta_y \varphi_k = \mu_k \varphi_k$  and  $\varphi_0 = \text{const.}$  We can write the following development

$$(3.36) \quad w_j(x, y) = \sum_{k \in \mathbb{N} \setminus \{0\}} a_{j,k}(x) \varphi_k(y)$$

(where the eigenfunction  $\varphi_0$  does not enter in the development). By using the representation in (3.36) we get

$$(3.37) \quad I_j \geq \sum_{k \neq 0} (\lambda_j - 1) |\mu_k|^2 \int_{\mathbb{R}_x^n} |a_{j,k}(x)|^2 dx - (\alpha + 1) \sum_{k \neq 0} \int_{\mathbb{R}_x^n} |u_{\bar{\omega}}(x)|^\alpha |a_{j,k}(x)|^2 dx$$

and by (3.23) we get

$$(3.38) \quad III_j = o(1) \|w_j\|_{L_{x,y}^2}^2$$

By combining (3.37) with (3.38) we get

$$(3.39) \quad I_j + III_j \geq 0$$

for  $j$  large enough. In order to estimate  $II_j$  notice that by the Cauchy-Schwartz inequality we get

$$(3.40) \quad \left| \int_{M_y^k} \int_{\mathbb{R}_x^n} \sqrt{-\Delta_y} (u_{\lambda_j} ((\alpha + 1) |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j dx dv_{M_y^k} \right| \\ \leq \left\| \sqrt{-\Delta_y} (u_{\lambda_j} ((\alpha + 1) |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha)) \right\|_{L_x^{\frac{2(n+k)}{n+k+2}} L_y^{\frac{2(n+k)}{n+k+2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

$$\leq C \|\nabla_y(u_{\lambda_j}((\alpha+1)|u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha))\|_{L_x^{\frac{2(n+k)}{n+k+2}} L_y^{\frac{2(n+k)}{n+k+2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

where at the last step we have used the following estimate

$$(3.41) \quad \forall p \in (1, \infty) \exists c(p), C(p) > 0 \text{ s.t.}$$

$$c(p) \|\sqrt{-\Delta_y} f\|_{L_y^p} \leq \|\nabla_y f\|_{L_y^p} \leq C(p) \|\sqrt{-\Delta_y} f\|_{L_y^p}$$

Indeed, using [10, Theorem 3.3.1], we have that  $\sqrt{-\Delta_y}$  is a first order classical pseudo differential operator on  $M$  with a principle symbol  $(g^{i,j}(y)\xi_i\xi_j)^{1/2}$ . Observe that

$$C_1 \sum_{i,j} g^{i,j}(y)\xi_i\xi_j \leq \sum_i \left| \sum_j g^{i,j}(y)\xi_j \right|^2 \leq C_2 |\xi|^2 \leq C_3 \sum_{i,j} g^{i,j}(y)\xi_i\xi_j$$

Moreover one can assume that in (3.41)  $f$  has no zero frequency. Then one can deduce (3.41) by working in local coordinates, introducing a classical angular partition of unity according to the index  $l \in [1, \dots, k]$  such that

$$\sum_{i,j} g^{i,j}(y)\xi_i\xi_j \leq c \left| \sum_j g^{l,j}(y)\xi_j \right|^2$$

and, most importantly, using the  $L^p$  boundedness of zero order pseudo differential operators on  $\mathbb{R}^k$  (for the proof of this fact we refer to [10, Theorem 3.1.6]).

Next, by the chain rule we get

$$\begin{aligned} & \nabla_y \left( u_{\lambda_j}((\alpha+1)|u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha) \right) \\ &= (\alpha+1) \nabla_y u_{\lambda_j} \left( |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha \right) \end{aligned}$$

and by the Hölder inequality we can continue the estimate (3.40) as follows

$$\dots \leq C \left\| \|\nabla_y u_{\lambda_j}\|_{L_y^q} \| |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L_y^r} \right\|_{L_x^{\frac{2(n+k)}{n+k+2}} L_{x,y}^{\frac{2(n+k)}{n+k-2}}} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

where

$$\frac{1}{q} + \frac{1}{r} = \frac{n+k+2}{2(n+k)}$$

and again by the Hölder inequality in the  $x$ -variable we can continue

$$\dots \leq C \|\nabla_y u_{\lambda_j}\|_{L_{x,y}^q} \| |u_{\bar{\omega}}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L_{x,y}^r} \|w_j\|_{L_{x,y}^{\frac{2(n+k)}{n+k-2}}}$$

Notice that if we fix

$$q = \frac{2(n+k)}{n+k-2} \text{ and } r = \frac{n+k}{2}$$

then by combining the Sobolev embedding

$$(3.42) \quad H_{x,y}^1 \subset L_{x,y}^{\frac{2(n+k)}{n+k-2}}$$

with (3.32) and (3.41), we can continue the estimate

$$\dots \leq o(1) \|\sqrt{-\Delta_y} u_{\lambda_j}\|_{L_{x,y}^q} \|w_j\|_{H_{x,y}^1} = o(1) \|w_j\|_{H_{x,y}^1}^2$$

where  $\lim_{j \rightarrow \infty} o(1) = 0$ . By combining this information in conjunction with the structure of  $II_j$  we get

$$(3.43) \quad II_j \geq \|w_j\|_{H_{x,y}^1}^2 (1 - o(1)) \geq 0 \text{ for } j > j_0$$

By combining (3.35), (3.39) and (3.43) we deduce  $w_j = 0$  for  $j$  large enough.

□

**Proof of Theorem 3.1** By using the diamagnetic inequality we deduce that (up to a remodulation factor  $e^{i\theta}$ ) we can assume that  $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$  is real valued. Moreover if  $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$  then also  $|v| \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ . By a standard application of the strong maximum principle we finally deduce that it is not restrictive to assume that  $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$  and  $v(x, y) > 0$ ,  $\forall (x, y) \in \mathbb{R}_x^n \times M_y^k$ .

*First step:*  $\exists \tilde{\lambda} > 0$  s.t.  $\forall v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ ,  $v(x, y) > 0$  we have  $\nabla_y v = 0$ ,  $\forall \lambda > \tilde{\lambda}$

Assume that the conclusion is false then there exists  $\lambda_j \rightarrow \infty$  such that  $u_{\lambda_j}(x, y) \in \mathcal{M}_{n,M^k,\alpha,\lambda_j}$ ,  $u_{\lambda_j}(x, y) > 0$  and  $\nabla_y u_{\lambda_j} \neq 0$ . This is absurd due to Lemma 3.5.

*Second step: conclusion*

We define

$$\lambda^* = \inf_{\lambda} \{ \lambda > 0 \mid \nabla_y v = 0 \ \forall v \in \mathcal{M}_{n,M^k,\alpha,\lambda} \}$$

By the first step  $\lambda^* < \infty$ . Moreover it is easy to deduce that if  $\lambda > \lambda^*$  then the minimizers of the problem  $J_{n,M^k,\alpha,\lambda}$  are precisely the same minimizers of the problem  $I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$ , which in turn are characterized in section 2 (hence we get (3.2)).

Next we prove that  $\lambda^* > 0$ . It is sufficient to show that

$$(3.44) \quad \lim_{\lambda \rightarrow 0} J_{n,M^k,\alpha,\lambda} < \text{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$$

(see (2.1) and (3.1) for a definition of the quantities involved in the inequality above). Let us fix  $\rho(y) \in C^\infty(M^k)$  such that

$$\int_{M^k} |\rho|^2 d\text{vol}_{M_y^k} = 1$$

and  $\rho^2(y_0) \neq \frac{1}{\text{vol}(M^k)}$  for some  $y_0 \in M^k$  (i.e.  $\rho(y)$  is not identically constant). Then we introduce the functions

$$\psi(x, y) = \rho(y)^{4/(4-\alpha n)} Q(\rho(y)^{(2\alpha)/(4-\alpha n)} x)$$

where  $Q(x)$  is the unique radially symmetric minimizer for  $I_{n,\alpha}^{1/\sqrt{\text{vol}(M^k)}}$ . Then we get

$$\|\psi(x, y)\|_{L_x^2}^2 = (\rho(y))^2 \text{ and } \mathcal{E}_{n,\alpha}(\psi(x, y)) = I_{n,\alpha}^1(\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}}$$

and as a consequence we deduce

$$\begin{aligned} & \int_{M_y^k} \int_{\mathbb{R}_x^n} \left( \frac{1}{2} |\nabla_x \psi(x, y)|^2 - \frac{1}{2+\alpha} |\psi(x, y)|^{2+\alpha} \right) dx d\text{vol}_{M_y^k} \\ &= I_{n,\alpha}^1 \int_{M_y^k} (\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}} d\text{vol}_{M_y^k} \\ &< I_{n,\alpha}^1 \left( \int_{M^k} (\rho(y))^2 d\text{vol}_{M_y^k} \right)^{\frac{4-\alpha n+2\alpha}{4-\alpha n}} \text{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}} = I_{n,\alpha}^1 \text{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}} \end{aligned}$$

where at the last inequality we have used the fact that  $I_{n,\alpha}^1 < 0$  in conjunction with the Hölder inequality (moreover we get the inequality  $<$  since by hypothesis  $\rho(y)$  is not identically constant). As a byproduct we get

$$\lim_{\lambda \rightarrow 0} \mathcal{E}_{n,M^k,\alpha,\lambda}(\psi(x,y)) < I_{n,\alpha}^1 \text{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}} = \text{vol}(M^k) I_{n,\alpha}^1 / \sqrt{\text{vol}(M^k)}$$

(where we have used (2.3)) which in turn implies (3.44).

Let us finally prove (3.3). It is sufficient to show that if  $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$  for  $\lambda < \lambda^*$  then  $\nabla_y v \neq 0$ . Assume by the absurd that it is false, then we get  $\lambda_1 < \lambda^*$  and  $v_1 \in \mathcal{M}_{n,M^k,\alpha,\lambda_1}$  such that  $\nabla_y v_1 = 0$ . Arguing as above it implies that

$$(3.45) \quad J_{n,M^k,\alpha,\lambda_1} = \text{vol}(M^k) I_{n,\alpha}^1 / \sqrt{\text{vol}(M^k)}$$

On the other hand by definition of  $\lambda^*$  there exists  $\lambda_2 \in (\lambda_1, \lambda^*]$  and  $v_2 \in \mathcal{M}_{n,M^k,\alpha,\lambda_2}$  such that  $\nabla_y v_2 \neq 0$ . As a consequence we deduce that

$$J_{n,M^k,\alpha,\lambda_1} < \mathcal{E}_{n,M^k,\alpha,\lambda_2}(v_2) = J_{n,M^k,\alpha,\lambda_2} \leq \text{vol}(M^k) I_{n,\alpha}^1 / \sqrt{\text{vol}(M^k)}$$

where at the last step we have used (3.7). Hence we get a contradiction with (3.45).  $\square$

#### 4. PROOF OF THEOREM 1.3

In the sequel the homogeneity of the euclidean space  $\mathbb{R}^n$  will play a key role. Due to this property we shall be able to reduce the proof of Theorem 1.3 to the problem studied in the previous section.

In view of section 2 it is sufficient to prove that there exists  $\rho^* > 0$  such that

$$(4.1) \quad v \in \mathcal{M}_{n,M^k,\alpha}^\rho \text{ implies } \nabla_y v = 0 \text{ for } \rho < \rho^*$$

and

$$(4.2) \quad v \in \mathcal{M}_{n,M^k,\alpha}^\rho \text{ implies } \nabla_y v \neq 0 \text{ for } \rho > \rho^*$$

By an elementary computation we have that the map

$$S_1 \ni u \rightarrow \rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y) \in S_\rho$$

where

$$S_\lambda = \{v \in H^1(\mathbb{R}^n \times M^k) \mid \|v\|_{L_{x,y}^2} = \lambda\}$$

is a bijection. Moreover we have

$$\begin{aligned} \mathcal{E}_{n,M^k,\alpha}(\rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y)) &= \rho^{(8-2\alpha n)/(4-\alpha n)} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u|^2 dx d\text{vol}_{M_y^k} \\ &\quad + \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u|^2 dx d\text{vol}_{M_y^k} \\ &\quad - \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \frac{1}{2+\alpha} \int_{M_y^k} \int_{\mathbb{R}_x^n} |u|^{2+\alpha} dx d\text{vol}_{M_y^k} \\ &= \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \left( \frac{1}{2} \rho^{-4\alpha/(4-\alpha n)} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y u|^2 dx d\text{vol}_{M_y^k} \right. \\ &\quad \left. + \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x u|^2 - \frac{1}{2+\alpha} |u|^{2+\frac{4}{\alpha}} dx d\text{vol}_{M_y^k} \right) \end{aligned}$$

In particular (4.1) and (4.2) are satisfied provided that there exists  $\rho^* > 0$  such that

$$(4.3) \quad v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \text{ implies } \nabla_y v = 0 \text{ for } \rho < \rho^*$$

and

$$(4.4) \quad v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \text{ implies } \nabla_y v \neq 0 \text{ for } \rho > \rho^*$$

that in turn follow by Theorem 3.1.

## 5. PROOF OF THEOREM 1.4

The main tool we use is the following Strichartz type estimates (whose proof follows by [12]).

**Proposition 5.1.** *For every manifold  $M_y^k$ ,  $n \geq 1$  and  $p, q \in [2, \infty]$  such that:*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (p, n) \neq (2, 2)$$

*there exists  $C > 0$  such that*

$$(5.1) \quad \|e^{-it\Delta_{x,y}} f\|_{L_t^p L_x^q H_y^1} + \left\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \right\|_{L_t^p L_x^q H_y^1} \\ \leq C \left( \|f\|_{L_x^2 H_y^1} + \|F\|_{L_t^{p'} L_x^{q'} H_y^1} \right);$$

$$(5.2) \quad \|\nabla_x e^{-it\Delta_{x,y}} f\|_{L_t^p L_x^q L_y^2} + \left\| \nabla_x \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \right\|_{L_t^p L_x^q L_y^2} \\ \leq C \left( \|\nabla_x f\|_{L_x^2 L_y^2} + \|\nabla_x F\|_{L_t^{p'} L_x^{q'} L_y^2} \right)$$

*and*

$$(5.3) \quad \left\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \right\|_{L_t^p L_x^q L_y^2} \leq C \|F\|_{L_t^{p'} L_x^{q'} L_y^2}$$

*Moreover*

$$(5.4) \quad \|e^{-it\Delta_{x,y}} f\|_{L_t^\infty L_x^2 H_y^1} + \left\| \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \right\|_{L_t^\infty L_x^2 H_y^1} \\ \leq C \left( \|f\|_{L_x^2 H_y^1} + \|F\|_{L_t^{p'} L_x^{q'} H_y^1} \right)$$

*and*

$$(5.5) \quad \|\nabla_x e^{-it\Delta_{x,y}} f\|_{L_t^\infty L_x^2 L_y^2} + \left\| \nabla_x \int_0^t e^{-i(t-s)\Delta_{x,y}} F(s) ds \right\|_{L_t^\infty L_x^2 L_y^2} \\ \leq C \left( \|\nabla_x f\|_{L_x^2 L_y^2} + \|\nabla_x F\|_{L_t^{p'} L_x^{q'} L_y^2} \right)$$

Next we shall use the norms  $\|\cdot\|_{X_T}$  and  $\|\cdot\|_{Y_T}$  introduced in (1.15) and (1.16) for time dependent functions. We also introduce the space  $Z_T$  whose norm is defined by

$$\|v\|_{Z_T} \equiv \|v\|_{X_T} + \|v\|_{Y_T}$$

and the nonlinear operator associated to the Cauchy problem (1.1):

$$\mathcal{T}_\varphi(u) \equiv e^{-it\Delta_{x,y}} \varphi + \int_0^t e^{-i(t-s)\Delta_{x,y}} u(s) |u(s)|^\alpha ds$$



We split the proof of Theorem 1.4 in several steps.

**5.1. Local Well Posedness.** This subsection is devoted to the proof of the following fact:

$$\begin{aligned} \forall \varphi \in H^1(\mathbb{R}^n \times M^1) \quad \exists T = T(\|\varphi\|_{H^1(\mathbb{R}^n \times M^1)}) > 0 \\ \text{and } \exists! v(t, x) \in Z_T \cap \mathcal{C}((-T, T); H^1(\mathbb{R}^n \times M^1)) \\ \text{s.t. } \mathcal{T}_\varphi v(t) = v(t) \quad \forall t \in (-T, T) \end{aligned}$$

*First step:*

$$\begin{aligned} \forall \varphi \in H^1(\mathbb{R}^n \times M^1) \quad \exists T = T(\|\varphi\|_{H^1(\mathbb{R}^n \times M^1)}) > 0, R = R(\|\varphi\|_{H^1(\mathbb{R}^n \times M^1)}) > 0 \text{ s.t.} \\ \mathcal{T}_\varphi(B_{Z_{\tilde{T}}}(0, R)) \subset B_{Z_{\tilde{T}}}(0, R) \quad \forall \tilde{T} < T \end{aligned}$$

First we estimate the nonlinear term:

$$\|u|u|^\alpha\|_{L_t^{p'} L_x^{q'} H_y^1} \leq \left\| \|u^\alpha(t, x, \cdot)\|_{L_y^\infty} \|u(t, x, \cdot)\|_{H_y^1} \right\|_{L_t^{p'} L_x^{q'}}$$

(where  $(p, q)$  is the couple in (1.15) and (1.16)) and after application of the Hölder inequality in  $(t, x)$  we get

$$\begin{aligned} \dots &\leq \|u\|_{L_t^p L_x^q H_y^1} \|u\|_{L_t^{\alpha \tilde{p}} L_x^{\alpha \tilde{q}} L_y^\infty}^\alpha \\ &\leq C \|u\|_{L_t^p L_x^q H_y^1} \|u\|_{L_t^{\alpha \tilde{p}} L_x^{\alpha \tilde{q}} H_y^1}^\alpha \end{aligned}$$

where we have used the embedding  $H_y^1 \subset L_y^\infty$  and we have chosen

$$\begin{aligned} \frac{1}{\tilde{p}} + \frac{1}{p} &= 1 - \frac{1}{p} \\ \frac{1}{\tilde{q}} + \frac{1}{q} &= 1 - \frac{1}{q} \end{aligned}$$

By direct computation we have:

$$(5.6) \quad \alpha \tilde{q} = q \text{ and } \alpha \tilde{p} < p$$

By combining the nonlinear estimate above with (5.1), (5.6) and the Hölder inequality (in the time variable) we get:

$$(5.7) \quad \|\mathcal{T}_\varphi u\|_{X_T} \leq C(\|\varphi\|_{L_x^2 H_y^1} + T^{a(d)} \|u\|_{X_T}^{1+\alpha})$$

with  $a(d) > 0$ .

Arguing as above get

$$\begin{aligned} \|\nabla_x(u|u|^\alpha)\|_{L_t^{p'} L_x^{q'} L_y^2} &\leq C \|\nabla_x u\|_{L_t^p L_x^q L_y^2} \|u^\alpha\|_{L_t^{\tilde{p}} L_x^{\tilde{q}} L_y^\infty} \\ &\leq C \|u\|_{Y_T} \|u\|_{L_t^{\alpha \tilde{p}} L_x^{\alpha \tilde{q}} H_y^1}^\alpha \end{aligned}$$

where  $\tilde{p}$  and  $\tilde{q}$  are as above and we have used the embedding  $H_y^1 \subset L_y^\infty$ . As a consequence of this estimate and (5.2) we get:

$$(5.8) \quad \|\mathcal{T}_\varphi u\|_{Y_T} \leq C(\|\nabla_x \varphi\|_{L_{x,y}^2} + T^{a(d)} \|u\|_{Y_T} \|u\|_{X_T}^\alpha)$$

with  $a(d) > 0$ .

By combining (5.7) with (5.8) we get

$$\|\mathcal{T}_\varphi u\|_{Z_T} \leq C(\|\varphi\|_{H^1(\mathbb{R}^n \times M^1)} + T^{a(d)} \|u\|_{Z_T} \|u\|_{Z_T}^\alpha)$$

The proof follows by a standard continuity argument.

Next we introduce the norm

$$\|w(t, x, y)\|_{\tilde{Z}_T} \equiv \|w(t, x, y)\|_{L^p((-T, T); L_x^q L_y^2)}$$

and we shall prove the following.

*Second step: let  $T, R > 0$  as in the previous step then*

$$\exists T' = T'(\|\varphi\|_{H^1(\mathbb{R}^n \times M^1)}) < T \text{ s.t. } \mathcal{T}_\varphi$$

*is a contraction on  $B_{Z_{T'}}(0, R)$  endowed with the norm  $\|\cdot\|_{\tilde{Z}_{T'}}$*

It is sufficient to prove:

$$(5.9) \quad \|\mathcal{T}_\varphi v_1 - \mathcal{T}_\varphi v_2\|_{\tilde{Z}_T} \leq CT^{a(d)} \|v_1 - v_2\|_{\tilde{Z}_T} \sup_{i=1,2} \{\|v_i\|_{Z_T}\}^\alpha$$

with  $a(d) > 0$ . Notice that we have

$$\begin{aligned} & \|v_1|v_1|^\alpha - v_2|v_2|^\alpha\|_{L^{p'}((-T, T); L_x^{q'} L_y^2)} \\ & \leq C \left\| \|v_1 - v_2\|_{L_y^2} (\|v_1\|_{L_y^\infty} + \|v_2\|_{L_y^\infty})^\alpha \right\|_{L^{p'}((-T, T); L_x^{q'})} \\ & \leq CT^{a(d)} \|v_1 - v_2\|_{\tilde{Z}_T} \sup_{i=1,2} \{\|v_i\|_{Z_T}\}^\alpha \end{aligned}$$

where we have used the Sobolev embedding  $H_y^1 \subset L_y^\infty$  and the Hölder inequality in the same spirit as in the proof of (5.7) and (5.8). We conclude by combining the estimate above with the Strichartz estimate (5.3).

*Third step: existence and uniqueness of solution in  $Z_{T'}$  where  $T'$  is as in the previous step*

We apply the contraction principle to the map  $\mathcal{T}_\varphi$  defined on the complete space  $B_{Z_{T'}}(0, R)$  endowed with the topology induced by  $\|\cdot\|_{\tilde{Z}_{T'}}$ . It is well-known that this space is complete.

*Fourth step: regularity of the solution*

By combining the previous steps with the fixed point argument we get the existence of a solution  $v \in Z_{T'}$ . In order to get the regularity  $v \in \mathcal{C}((-T', T'); H^1(\mathbb{R}^n \times M^1))$  it is sufficient to argue as in the first step (to estimate the nonlinearity) in conjunction with the Strichartz estimates (5.4) and (5.5).

**5.2. Global Well Posedness.** Next we prove that the local solution (whose existence has been proved above) cannot blow-up in finite time. The argument is standard and follows from the conservation laws:

$$(5.10) \quad \|u(t)\|_{L_{x,y}^2} \equiv \|\varphi\|_{L_{x,y}^2}$$

$$(5.11) \quad \mathcal{E}_{n, M^1, \alpha}(u(t)) + \frac{1}{2} \|u(t)\|_{L_{x,y}^2}^2 \equiv \mathcal{E}_{n, M^1, \alpha}(\varphi) + \frac{1}{2} \|\varphi\|_{L_{x,y}^2}^2$$

where  $\mathcal{E}_{n,M^1,\alpha}$  is defined in (1.2). By the Gagliardo Nirenberg inequality we deduce

$$\mathcal{E}_{n,M^1,\alpha}(u(t)) + \frac{1}{2}\|u(t)\|_{L_{x,y}^2}^2 \geq \frac{1}{2}\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}^2 - C\|u(t)\|_{L_{x,y}^2}^{2+\alpha-\mu}\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}^\mu$$

for a suitable  $\mu \in (0, 2)$ . By combining the estimate above with (5.10) and (5.11) we get

$$\frac{1}{2}\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}^2 - C\|\varphi\|_{L_{x,y}^2}^{2+\alpha-\mu}\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}^\mu \leq \mathcal{E}_{n,M^1,\alpha}(\varphi) + \frac{1}{2}\|\varphi\|_{L_{x,y}^2}^2$$

Since  $\mu \in (0, 2)$  it implies that  $\|u(t)\|_{H^1(\mathbb{R}^n \times M^1)}$  cannot blow-up in finite time.

## 6. APPENDIX

For the sake of completeness we prove in this appendix Theorems 1.1 and 1.2. Our argument is heavily inspired by the work [5] even if, in our opinion, the following presentation of Theorem 1.1 is simpler compared with the original one.

**Proof of Theorem 1.1** For any given  $\rho > 0$  we shall denote by  $u_{j,\rho} \in H^1(\mathbb{R}^n \times M^k)$  any constrained minimizing sequence, i.e.:

$$(6.1) \quad \|u_{j,\rho}\|_{L_{x,y}^2} = \rho \text{ and } \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}) = K_{n,M^k,\alpha}^\rho$$

Next we split the proof in many steps.

*First step:*  $K_{n,M^k,\alpha}^\rho > -\infty$  and  $\sup_j \|u_{j,\rho}\|_{H_{x,y}^1} < \infty$ ,  $\forall \rho > 0$

By the classical Gagliardo Nirenberg inequality (see (1.4)) we get the existence of  $\mu \in (0, 2)$  such that

$$\begin{aligned} & \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}) + \frac{1}{2}\rho^2 \\ & \geq \frac{1}{2} \int_{M_y^k} \int_{\mathbb{R}^n} (|\nabla_{x,y} u_{j,\rho}|^2 + |u_{j,\rho}|^2) dx d\text{vol} M_y^k - C(\rho) \|u_{j,\rho}\|_{H^1(\mathbb{R}^n \times M^k)}^\mu \\ & \geq \inf_{t>0} (1/2t^2 - C(\rho)t^\mu) > -\infty \end{aligned}$$

The conclusion follows by a standard argument.

*Second step:* the map  $(0, \infty) \ni \rho \rightarrow K_{n,M^k,\alpha}^\rho$  is continuous

Fix  $\rho \in (0, \infty)$  and let  $\rho_j \rightarrow \rho$ . Then we have

$$\begin{aligned} K_{n,M^k,\rho}^{\rho_j} & \leq \mathcal{E}_{n,M^k,\alpha}\left(\frac{\rho_j}{\rho} u_{j,\rho}\right) = \\ & \left(\frac{\rho_j}{\rho}\right)^2 \left( \frac{1}{2} \|\nabla_{x,y} u_{j,\rho}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \left(\frac{\rho_j}{\rho}\right)^\alpha \|u_{j,\rho}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \right) \\ & = \left(\frac{\rho_j}{\rho}\right)^2 \left( \frac{1}{2} \|\nabla_{x,y} u_{j,\rho}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \|u_{j,\rho}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \right) \\ & \quad + \frac{1}{2+\alpha} \left(\frac{\rho_j}{\rho}\right)^2 \left( 1 - \left(\frac{\rho_j}{\rho}\right)^\alpha \right) \|u_{j,\rho}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \\ & = \left( \frac{1}{2} \|\nabla_{x,y} u_{j,\rho}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \|u_{j,\rho}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \right) \end{aligned}$$

$$\begin{aligned}
& + \left( \left( \frac{\rho_j}{\rho} \right)^2 - 1 \right) \left( \frac{1}{2} \|\nabla_{x,y} u_{j,\rho}\|_{L^2_{x,y}}^2 - \frac{1}{2+\alpha} \|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}}^{2+\alpha} \right) \\
& + \frac{1}{2+\alpha} \left( \frac{\rho_j}{\rho} \right)^2 \left( 1 - \left( \frac{\rho_j}{\rho} \right)^\alpha \right) \|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}}^{2+\alpha}
\end{aligned}$$

Since we are assuming that  $\rho_j \rightarrow \rho$  and  $\sup_n \|u_{j,\rho}\|_{H^1(\mathbb{R}^n \times M^k)} < \infty$  (see the first step) we get

$$\limsup_{j \rightarrow \infty} K_{n,M^k,\alpha}^{\rho_j} \leq K_{n,M^k,\alpha}^\rho$$

To prove the opposite inequality let us fix  $u_j \in H^1(\mathbb{R}^n \times M^k)$  such that

$$(6.2) \quad \|u_j\|_{L^2_{x,y}} = \rho_j \text{ and } \mathcal{E}_{n,M^k,\alpha}(u_j) < K_{n,M^k,\alpha}^{\rho_j} + \frac{1}{j}$$

By looking at the proof of the first step we also deduce that  $u_j$  can be chosen in such a way that

$$(6.3) \quad \sup_j \|u_j\|_{H^1(\mathbb{R}^n \times M^k)} < \infty$$

Then we can argue as above and we get

$$\begin{aligned}
K_{n,M^k,\alpha}^\rho & \leq \mathcal{E}_{n,M^k,\alpha} \left( \frac{\rho}{\rho_j} u_j \right) \\
& = \left( \frac{1}{2} \|\nabla_{x,y} u_j\|_{L^2_{x,y}}^2 - \frac{1}{2+\alpha} \|u_j\|_{L^{2+\alpha}_{x,y}}^{2+\alpha} \right) \\
& + \left( \left( \frac{\rho}{\rho_j} \right)^2 - 1 \right) \left( \frac{1}{2} \|\nabla_{x,y} u_j\|_{L^2_{x,y}}^2 - \frac{1}{2+\alpha} \|u_j\|_{L^{2+\alpha}_{x,y}}^{2+\alpha} \right) \\
& + \frac{1}{2+\alpha} \left( \frac{\rho}{\rho_j} \right)^2 \left( 1 - \left( \frac{\rho}{\rho_j} \right)^\alpha \right) \|u_j\|_{L^{2+\alpha}_{x,y}}^{2+\alpha}
\end{aligned}$$

By using (6.2), (6.3) and the assumption  $\rho_j \rightarrow \rho$  we get

$$K_{n,M^k,\alpha}^\rho \leq \liminf_{j \rightarrow \infty} K_{n,M^k,\alpha}^{\rho_j}$$

*Third step: for every  $\rho > 0$  we have (up to subsequence)  $\inf_j \|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}} > 0$*

It is sufficient to prove that  $K_{n,M^k,\alpha}^\rho < 0$ . In fact we have

$$(6.4) \quad K_{n,M^k,\alpha}^\rho \leq \text{vol}(M^k) \mathcal{E}_{n,\alpha}(u_{n,\omega,\alpha}) = \text{vol}(M^k) I_{n,\alpha}^\rho / \sqrt{\text{vol}(M^k)} < 0$$

where  $\mathcal{E}_{n,\alpha}$  is the energy defined in (2.2) and  $\omega$  is chosen in such a way that  $\|u_{n,\omega,\alpha}\|_{L^2_x} = \frac{\rho}{\sqrt{\text{vol}(M^k)}}$ . Notice that in (6.4) we have used (2.4) and (2.5).

*Fourth step: for any minimizing sequence  $u_{j,\rho}$  there exists  $\tau_j \in \mathbb{R}^n$  s.t. (up to subsequence)  $u_{j,\rho}(x + \tau_j, y)$  has a weak limit  $\bar{u} \neq 0$*

We have the following localized Gagliardo Nirenberg inequality:

$$(6.5) \quad \|v\|_{L^{2+4/(n+k)}_{x,y}} \leq C \sup_{x \in \mathbb{R}^n} \left( \|v\|_{L^2_{Q_x^n \times M^k}} \right)^{2/(n+k+2)} \|v\|_{H^1(\mathbb{R}^n \times M^k)}^{(n+k)/(n+k+2)}$$

where

$$Q_x^n = x + [0, 1]^n \quad \forall x \in \mathbb{R}^n$$

The estimate above can be proved as follows (see [8] for a similar argument on the flat space  $\mathbb{R}^{d+k}$ ). We fix  $x_h \in \mathbb{R}^n$  in such a way that  $\bigcup_h Q_{x_h}^n = \mathbb{R}^n$  and  $meas_n(Q_{x_i}^n \cap Q_{x_j}^n) = 0$  for  $i \neq j$  where  $meas_n$  denotes the Lebesgue measure in  $\mathbb{R}^n$ . By the classical Gagliardo Nirenberg inequality we get:

$$\|v\|_{L_{Q_{x_h}^n \times M^k}^{2+4/(n+k)}}^{2+4/(n+k)} \leq C \|v\|_{L_{Q_{x_h}^n \times M^k}^2}^{4/(n+k)} \|v\|_{H^1(Q_{x_h}^n \times M^k)}^2$$

The proof of (6.5) follows by taking the sum of the previous estimates on  $h \in \mathbb{N}$ . Due to the boundedness of  $u_{j,\rho}$  in  $H^1(\mathbb{R}^m \times M^k)$  (see the first step) we deduce by (6.5) that

$$(6.6) \quad 0 < \epsilon_0 = \inf_j \|u_{j,\rho}\|_{L_{x,y}^{2+4/(n+k)}} \leq C \sup_{x \in \mathbb{R}^n} \|u_{j,\rho}\|_{L_{Q_{x_h}^n \times M^k}^2}^{2/(n+k+2)}$$

(the l.h.s. above follows by combining the Hölder inequality with the third step). The proof can be concluded by the Rellich compactness theorem once we choose a sequence  $\tau_j \in \mathbb{R}_x^n$  in such a way that

$$\inf_j \|u_{j,\rho}\|_{L_{Q_{\tau_j}^n \times M^k}^2} > 0$$

(the existence of such a sequence  $\tau_j$  follows by (6.6)).

*Fifth step: the map  $(0, \bar{\rho}) \ni \rho \rightarrow \rho^{-2} K_{n,M^k,\alpha}^\rho$  is strictly decreasing*

Let us fix  $\rho_1 < \rho_2$  and  $u_{j,\rho_1}$  a minimizing sequence for  $K_{n,M^k,\alpha}^{\rho_1}$ . Then we have

$$\begin{aligned} K_{n,M^k,\alpha}^{\rho_2} &\leq \mathcal{E}_{n,M^k,\alpha}\left(\frac{\rho_2}{\rho_1} u_{j,\rho_1}\right) \\ &= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2} \|\nabla_{x,y} u_{j,\rho_1}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \left(\frac{\rho_2}{\rho_1}\right)^\alpha \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha}\right) \\ &= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2} \|\nabla_{x,y} u_{j,\rho_1}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha}\right) \\ &\quad + \frac{1}{2+\alpha} \left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^\alpha\right) \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \\ &\leq \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2} \|\nabla_{x,y} u_{j,\rho_1}\|_{L_{x,y}^2}^2 - \frac{1}{2+\alpha} \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha}\right) \\ &\quad + \frac{1}{2+\alpha} \left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^\alpha\right) \inf_j \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} \end{aligned}$$

By recalling (see the third step) that  $\inf_j \|u_{j,\rho_1}\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} > 0$  we get

$$K_{n,M^k,\alpha}^{\rho_2} < \left(\frac{\rho_2}{\rho_1}\right)^2 K_{n,M^k,\alpha}^{\rho_1}$$

*Sixth step: let  $\bar{u}$  be as in the fourth step, then  $\|\bar{u}\|_{L_{x,y}^2} = \rho$*

Up to subsequence we get:

$$u_{j,\rho}(x + \tau_j, y) \rightarrow \bar{u}(x, y) \neq 0 \text{ a.e. } (x, y) \in \mathbb{R}_x^n \times M_y^k$$

and hence by the Brezis-Lieb lemma (see [1]) we get

$$(6.7) \quad \|u_{j,\rho}(x + \tau_j, y) - \bar{u}(x, y)\|_{L_{x,y}^{2+\alpha}}^{2+\alpha}$$

$$= \|u_{j,\rho}(x + \tau_j, y)\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} - \|\bar{u}(x, y)\|_{L_{x,y}^{2+\alpha}}^{2+\alpha} + o(1)$$

Assume that  $\|\bar{u}\|_{L_{x,y}^2} = \theta$ , our aim is to prove  $\theta = \rho$ . Since  $\bar{u} \neq 0$  necessarily  $\theta > 0$ . Notice that since  $L_{x,y}^2$  is an Hilbert space we have

$$(6.8) \quad \begin{aligned} \rho^2 &= \|u_{j,\rho}(x + \tau_j, y)\|_{L_{x,y}^2}^2 \\ &= \|u_{j,\rho}(x + \tau_j, y) - \bar{u}(x, y)\|_{L_{x,y}^2}^2 + \|\bar{u}(x, y)\|_{L_{x,y}^2}^2 + o(1) \end{aligned}$$

and hence

$$(6.9) \quad \|u_{j,\rho}(x + \tau_j, y) - \bar{u}(x, y)\|_{L_{x,y}^2}^2 = \rho^2 - \theta^2 + o(1)$$

By a similar argument

$$(6.10) \quad \begin{aligned} &\int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_x(u_{j,\rho}(x + \tau_j, y)) - \nabla_x \bar{u}(x, y)|^2 dx dy \\ &+ \int_{M_y^k} \int_{\mathbb{R}_x^n} |\nabla_y(u_{j,\rho}(x + \tau_j, y)) - \nabla_y \bar{u}(x, y)|^2 dx dvol_{M_y^k} \\ &+ \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_x \bar{u}(x, y)|^2 + |\nabla_y \bar{u}(x, y)|^2) dx dvol_{M_y^k} \\ &= \int_{M_y^k} \int_{\mathbb{R}_x^n} (|\nabla_x(u_{j,\rho}(x + \tau_j, y))|^2 + |\nabla_y u_{j,\rho}(x + \tau_j, y)|^2) dx dvol_{M_y^k} + o(1) \end{aligned}$$

By combining (6.10) with (6.7) we get:

$$(6.11) \quad \begin{aligned} K_{n,M^k,\alpha}^\rho &= \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}(x + \tau_j, y)) = \\ &\lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(u_{j,\rho}(x + \tau_j, y) - \bar{u}(x, y)) + \mathcal{E}_{n,M^k,\alpha}(\bar{u}) \end{aligned}$$

and we can continue the estimate as follows

$$\dots \geq K_{n,M^k,\alpha}^{\sqrt{\rho^2 - \theta^2} + o(1)} + K_{n,M^k,\alpha}^\theta$$

where we have used (6.9). Hence by using the second step we get

$$K_{n,M^k,\alpha}^\rho \geq K_{n,M^k,\alpha}^{\sqrt{\rho^2 - \theta^2}} + K_{n,M^k,\alpha}^\theta$$

Assume that  $\theta < \rho$ , then by using the monotonicity proved in fifth step we get

$$K_{n,M^k,\alpha}^\rho > \frac{\rho^2 - \theta^2}{\rho^2} K_{n,M^k,\alpha}^\rho + \frac{\theta^2}{\rho^2} K_{n,M^k,\alpha}^\rho = K_{n,M^k,\alpha}^\rho$$

and we have an absurd. □

**Proof of Theorem 1.2** Assume by the absurd that the conclusion is false, then there exists  $\rho$  and two sequences  $\varphi_j \in H^1(\mathbb{R}^n \times M^k)$  and  $t_j \in \mathbb{R}$  such that

$$(6.12) \quad \lim_{j \rightarrow \infty} \text{dist}_{H^1(\mathbb{R}^n \times M^k)}(\varphi_j, \mathcal{M}_{n,M^k,\alpha}^\rho) = 0$$

and

$$(6.13) \quad \liminf_{j \rightarrow \infty} \text{dist}_{H^1(\mathbb{R}^n \times M^k)}(u_{\varphi_j}(t_j), \mathcal{M}_{n,M^k,\alpha}^\rho) > 0$$

where  $u_{\varphi_j}$  is the solution to (1.1) with Cauchy data  $\varphi_j$ . By (6.12) we deduce the following informations:

$$\lim_{j \rightarrow \infty} \|\varphi_j\|_{L^2_{x,y}} = \rho \text{ and } \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(\varphi_j) = K_{n,M^k,\alpha}^\rho$$

and hence due to the conservation laws satisfied by solutions to (1.1) we get

$$\lim_{j \rightarrow \infty} \|u_{\varphi_j}(t_j)\|_{L^2_{x,y}} = \rho \text{ and } \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(u_{\varphi_j}(t_j)) = K_{n,M^k,\alpha}^\rho$$

In turn by an elementary computation we get:

$$\|\tilde{u}_j\|_{L^2_{x,y}} = \rho \text{ and } \lim_{j \rightarrow \infty} \mathcal{E}_{n,M^k,\alpha}(\tilde{u}_j) = K_{n,M^k,\alpha}^\rho$$

(more precisely  $\tilde{u}_j$  is constrained minimizing sequence for  $K_{n,M^k,\alpha}^\rho$ ) where

$$\tilde{u}_j = \rho \frac{u_{\varphi_j}(t_j)}{\|u_{\varphi_j}(t_j)\|_{L^2_{x,y}}}$$

Moreover by (6.13) it is easy to deduce

$$\liminf_{j \rightarrow \infty} \text{dist}_{H^1(\mathbb{R}^n \times M^k)}(\tilde{u}_j, \mathcal{M}_{n,M^k,\alpha}^\rho) > 0$$

and it is in contradiction with the compactness of minimizing sequences for  $K_{n,M^k,\alpha}^\rho$  stated in Theorem 1.1. □

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DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE CERGY-PONTOISE, 2, AVENUE ADOLPHE  
CHAUVIN, 95302 CERGY-PONTOISE CEDEX, FRANCE AND INSTITUT UNIVERSITAIRE DE FRANCE  
*E-mail address:* `nikolay.tzvetkov@u-cergy.fr`

UNIVERSITÀ DEGLI STUDI DI PISA DIPARTIMENTO DI MATEMATICA "L. TONELLI" LARGO BRUNO  
PONTECORVO 5 I - 56127 PISA. ITALY  
*E-mail address:* `viscigli@dm.unipi.it`